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Exceptional and non-crystallographic root systems and the Kochen–Specker theorem

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Abstract

The Kochen–Specker theorem states that a 3-dimensional complex Euclidean space admits a finite configuration of projective lines such that the corresponding quantum observables (the orthogonal projectors) cannot be assigned with 0 and 1 values in a classically consistent way. This paper shows that the irreducible root systems of exceptional and of non-crystallographic types are useful in constructing such configurations in other dimensions. The cases E_6 and E_7 lead to new examples, while F_4 , E_8 and H_4 yield a new interpretation of the known ones. The described configurations have an additional property: they are saturated, i.e. the tuples of mutually orthogonal lines, being partially ordered by inclusion, yield a poset with all maximal elements having the same cardinality (the dimension of space).

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1. Introduction

The aim of the present paper is to establish a link between several examples illustrating the Kochen–Specker theorem [1] (a result in non-relativistic quantum theory closely related to Bell’s inequalities) and the irreducible root systems (a notion emerging in the classification of finite-dimensional complex simple Lie algebras and of finite Coxeter groups).

Let us recall what the Kochen–Specker result is about. The main object is a *finite* collection $A \subset \mathbb{P}(\mathcal{H})$ of (projective) lines in a complex or real Hilbert space \mathcal{H} of *finite* dimension d . One is interested in the orthogonality relation \perp between the elements of A , and the aim is to assign to each $x \in A$ one of two colours, say *red* or *blue*, satisfying certain conditions. Every such (bi)colouring is described by a function $v : A \rightarrow \{0, 1\}$, where 0 corresponds, say, to the

blue colour and 1 corresponds to the red colour. Let us say that a bicolouring $v : A \rightarrow \{0, 1\}$ is *good* if (1) for all collections of mutually orthogonal $x_1, x_2, \dots, x_d \in A$ there exists a unique i_0 such that $v(x_{i_0}) = 1$; (2) $\forall x, y \in A$: if $y \perp x$ and $v(x) = 1$, then $y = 0$. S Kochen and E P Specker prove that if $d = 3$ and \mathcal{H} is complex, then there exists A which does not admit a good v (note that their construction is explicit and yields $|A| = 117$).

Let us call A *non-colourable* if it does not admit good bicolourings, and *colourable*—otherwise (leaving out the prefix *bi*). The motivation to look for such configurations comes from the analysis of quantum theory in terms of Bell’s inequalities meant to express the deviation of the behaviour of a physical system from a classical pattern. If each $x \in A$ is identified with a 0–1 observable (represented by the orthogonal projector on x), then the behaviour of the physical system with respect to the measurement acts of these observables will be extremely non-classical in the following sense. Take any A with the property mentioned. In classical physics an act of measurement of an observable is an act of *revealing* of its pre-existing value. Try to accept the same point of view in the quantum case, in particular, for the observables A . Suppose we have a collection of mutually orthogonal $x_1, x_2, \dots, x_k \in A$. Since the corresponding projectors commute, their observables can be measured simultaneously. If a simultaneous measurement act yields 1 for x_{i_0} , it yields 0 for the other $x_i, i \neq i_0$. If $k = d$, then x_{i_0} corresponding to 1 is always present. Therefore we can induce a *good* bicolouring v on A with $v(x)$ being the pre-existing value corresponding to $x \in A$. Since A *does not* admit such bicolourings, one may not interpret the acts of measurements with respect to A in a straightforward classical fashion.

The non-colourable configurations A are known to exist in every dimension $d \geq 3$. For $d \leq 2$ all projective configurations admit good (bi)colourings. It turns out that some of the examples have nice geometrical properties. If we look at the whole collection $\mathbb{P}(\mathcal{H}), \dim \mathcal{H} = d$, then we have the following: whenever $x_1, x_2, \dots, x_k \in \mathbb{P}(\mathcal{H}), k < d$, are mutually orthogonal, there exist $x_{k+1}, x_{k+2}, \dots, x_d$, such that x_1, x_2, \dots, x_d are mutually orthogonal. Let us require this property from a configuration $A \subset \mathbb{P}(\mathcal{H})$. Denote

$$\mathcal{P}_\perp^{(k)}(A) := \{U \subset A \mid \#U = k \ \& \ \forall x, y \in U : x \neq y \Rightarrow x \perp y\}.$$

Put $\mathcal{P}_\perp(A) := \cup_{k=0}^d \mathcal{P}_\perp^{(k)}(A)$. A is called *saturated* if $\forall U \in \mathcal{P}_\perp(A) \exists M \in \mathcal{P}_\perp^{(d)}(A)$ such that $M \supset U$. An easy example of a finite saturated configuration A in d dimensions is just a collection of d mutually orthogonal lines, but there exist much more complicated examples. Furthermore, there exist finite saturated configurations which do not admit good bicolourings! From a quantum-mechanical perspective, one may view such configurations as finite analogues of $\mathbb{P}(\mathcal{H})$.

Intuitively, a finite saturated projective configuration without good bicolourings is something very symmetric. This symmetry is essentially the subject of the present paper. It turns out that exceptional root systems and non-crystallographic root systems of finite Coxeter groups allow us to construct examples of such configurations. The idea is to consider the projective lines represented by the roots. There exist the following exceptional root systems: G_2, F_4, E_6, E_7 and E_8 . The non-crystallographic root systems are denoted by $I_2(p)$ ($p = 5$ or $p > 6$), H_3 and H_4 . Since a configuration is non-colourable only if $d \geq 3$, focus on the root systems F_4, E_6, E_7, E_8, H_3 and H_4 . The result is as follows.

Theorem 1.

- (1) *The finite projective configurations F_4, E_7, E_8 and H_4 are saturated and non-colourable.*
- (2) *The configuration H_3 is saturated, but colourable.*
- (3) *The configuration E_6 admits an extension up to a saturated finite configuration, which is non-colourable.*

The explicit description of the mentioned saturation of E_6 configuration will be given below (theorem 2). It is interesting to mention that it is realized by a presheaf-like construction that makes use of the remaining exceptional root system G_2 .

2. The root systems F_4 , E_8 , H_4 and H_3

The F_4 , E_8 and H_4 configurations correspond to the Kochen–Specker-type examples already considered in the literature. Therefore we make just a few remarks. The saturation property can be verified on a personal computer in a straightforward manner (for example, in Maple).

The projective configurations in \mathbb{R}^4 illustrating the Kochen–Specker theorem given by A Peres [2] (20 lines) and A Cabello, J M Esterbaranz, G García-Alcaine [3] (18 lines) can be viewed as subsets of the same set of 24 lines represented by the elements of the F_4 root system.

The root system E_8 is related to the Kochen–Specker-type example constructed by D Mermin [4] and by M Kernaghan, A Peres [5]. Their example involves 40 projective lines in \mathbb{R}^8 . The finite *saturated* configuration containing these lines has been constructed by A Ruuge, F Van Oystaeyen [6]. It consists of 120 projective lines. These lines can be viewed as projective lines corresponding to the 240 roots of the irreducible root system E_8 .

The H_4 case corresponds to the paper of P K Aravind, F Lee-Elkin [7]. There are 120 roots, which yield 60 projective lines in \mathbb{R}^4 .

The H_3 case is rather simple and does not yield a new example of a finite non-colourable configuration (in \mathbb{R}^3). The root system contains the vectors $(\pm 1, 0, 0)$, $(\pm 1, \pm 1/\tau, \pm \tau)$, plus all the vectors obtained from them by cyclic permutations of coordinates; here $\tau = (1 + \sqrt{5})/2$ is the *golden ratio* (recall that $\tau^2 = \tau + 1$). There are 30 roots in total, and therefore 15 projective lines. The corresponding configuration uniquely splits into five mutually disjoint triples of mutually orthogonal lines. It is saturated and colourable, admitting 3^5 good bicolourings.

3. The root system E_7

The case of the root system of the Coxeter group of type E_7 requires a little bit more work. It is necessary to make some remarks to persuade oneself in the fact that the corresponding (saturated) configuration (in \mathbb{R}^7) is not colourable.

It is convenient to model the E_7 root system (denote it by Φ) not on \mathbb{R}^7 , but on a 7-dimensional subspace of \mathbb{R}^8 consisting of all vectors (a_1, a_2, \dots, a_8) such that $\sum_{i=1}^8 a_i = 0$. One can obtain Φ by taking the union of the orbits of $(1, \bar{1}, 0, 0, 0, 0, 0)$ and $(1/2)(1, 1, 1, 1, \bar{1}, \bar{1}, \bar{1})$ under the natural action of S_8 on \mathbb{R}^8 ; here $\bar{1} = -1$. The result is $|\Phi| = 126$ and therefore we have 63 projective lines (rays). Let $[a_1, a_2, \dots, a_8]$ denote the ray represented by the vector (a_1, a_2, \dots, a_8) .

For each $k = 2, 3, \dots, 7$, one may consider all k -tuples of mutually orthogonal rays; denote the number of all such k -tuples by n_k . Then a (straightforward) Maple computation yields $n_2 = 945, n_3 = 4095, n_4 = 4725, n_5 = 2835, n_6 = 945, n_7 = 135$. It turns out that our configuration (63 rays) can be represented as a union of nine mutually disjoint 7-tuples of mutually orthogonal rays ($63 = 9 \times 7$). In fact, there are 960 possibilities of realizing such a disjoint union, but we are going to select just one of them. To describe it, it is convenient to index the components of $a \in \mathbb{R}^8$ not by $1, 2, \dots, 8$, but by the elements of the projective line $\mathbb{F}_7 \cup \{\infty\}$ over the field of seven elements, $a = (a_\infty, a_0, a_1, \dots, a_6)$.

Let k vary over \mathbb{F}_7 . Denote by $\lambda^{(k)}$ the ray represented by the vector $(a_\infty, a_0, \dots, a_6)$ having $a_\infty = 1, a_k = -1$, and all other components equal to 0. Denote by $\mu^{(k)}$ the ray represented by the vector having 1 at the positions $\infty, k, k + 1, k + 3$, and -1 at the other

four positions. Denote by $\nu^{(k)}$ the ray represented by the vector having 1 at the positions $\infty, k, k-1, k-3$ and -1 at the other four positions. Next, let $i \in \mathbb{F}_7$ vary over 1, 2, 3. Denote by $\xi^{(k,i)}$ the ray represented by the vector $(a_\infty, a_0, \dots, a_6)$ having $a_{k+i} = 1, a_{k-i} = -1$, and all other components equal to 0. Finally, denote by $\eta^{(k,i)}$ the ray represented by the vector having 1 at the positions $\infty, k, k+i, k-i$ and -1 at the other four positions.

With this notation we can describe the following 7-tuples of mutually orthogonal rays. Put $Q_k := \{\lambda^{(k)}, \xi^{(k,1)}, \xi^{(k,2)}, \xi^{(k,3)}, \eta^{(k,1)}, \eta^{(k,2)}, \eta^{(k,3)}\}, k \in \mathbb{F}_7$. Put $Q_+ := \{\mu^{(0)}, \mu^{(1)}, \dots, \mu^{(6)}\}$ and $Q_- := \{\nu^{(0)}, \nu^{(1)}, \dots, \nu^{(6)}\}$. It is straightforward to check that $Q_0, \dots, Q_6, Q_+, Q_-$ are mutually disjoint; their union is just the E_7 configuration. Let us also write Q_7 instead of Q_+ , and Q_8 instead of Q_- .

The verification that the configuration is non-colourable can now be completed on a computer. If it were colourable, one could choose in each Q_i an element l_i , in such a way that the rays l_1, l_2, \dots, l_9 were pairwise *non-orthogonal*. To verify that this is impossible, take any $x_1 \in Q_1$. Find $x_2 \in Q_2$ such that $x_2 \not\perp x_1$. After that, find $x_3 \in Q_3$ such that $x_3 \not\perp x_1$ and $x_3 \not\perp x_2$. If this is possible, try to find $x_4 \in Q_4$ such that $x_4 \not\perp x_1, x_2, x_3$, and so on. It turns out that one cannot reach this way the set Q_9 . Therefore the E_7 configuration is non-colourable. Introducing (in analogy with n_k) the numbers m_q for the numbers of all q -tuples of mutually *non-orthogonal* rays, one obtains $m_2 = 1008, m_3 = 5376, m_4 = 10080, m_5 = 8064, m_6 = 2016, m_7 = 288, m_8 = 0$. An example of seven mutually non-orthogonal rays is $\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(6)}$.

4. The root system E_6

This case is much more complicated than the other cases. The corresponding configuration does not contain tuples of pairwise orthogonal rays which have five or six elements. In particular, it is not saturated. It turns out that it is possible to construct a non-colourable saturated configuration containing it. Moreover, the construction makes use of the remaining G_2 root system, i.e. in the end *all* exceptional root systems turn out to be useful in constructing the examples of non-colourable saturated configurations. Generally speaking, what happens is that one computes all the 4-tuples of pairwise orthogonal lines in E_6 , and then attaches to each such tuple a copy of G_2 projective configuration. This presheaf-like construction turns out to be saturated. Some more lines are needed to achieve non-colourability, but the saturation property can be preserved.

Let us describe the roots of E_6 . It is convenient to model them on a 6-dimensional subspace R of \mathbb{R}^9 . A generic element of \mathbb{R}^9 is of the form $(x_1, x_2, x_3; y_1, y_2, y_3; z_1, z_2, z_3)$. It is convenient to use a shorter notation for it: $(x; y; z)$, where $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3)$, and $z = (z_1, z_2, z_3)$. The conditions defining R are $x_1 + x_2 + x_3 = 0, y_1 + y_2 + y_3 = 0$ and $z_1 + z_2 + z_3 = 0$.

The root system contains 72 elements. Some of the vectors are of the form $(\xi; \theta; \theta), (\theta; \xi; \theta)$ and $(\theta; \theta; \xi)$, where $\theta = (0, 0, 0)$, and $\xi = (1, \bar{1}, 0), (1, 0, \bar{1})$ or $(0, 1, \bar{1}), \bar{1} \equiv -1$. This yields nine elements. The other part of elements is given by the triples $(\xi; \eta; \zeta)$, where ξ, η, ζ vary over $\{(1/3)(2, \bar{1}, \bar{1}), (1/3)(\bar{1}, 2, \bar{1}), (1/3)(\bar{1}, \bar{1}, 2)\}$. This yields 27 elements; in total we obtain 36 elements. The remaining 36 elements of the root system are just the inverses of the described ones.

The 72 roots define 36 projective lines (rays). Denote this set by A . The configuration is quite symmetric: each line is orthogonal to precisely 15 other lines. Note that each line can be represented by an integer (9-dimensional) vector with the entries being 0, ± 1 or ± 2 . Denote by n_m the number of subsets $U \subset A$ consisting of m pairwise orthogonal lines. A computation in analogy with the E_7 case yields $n_2 = 270, n_3 = 540, n_4 = 135, n_5 = 0, n_6 = 0$. There

are no tuples of cardinality 5 and 6, but for smaller tuples one can check that each pair of orthogonal lines extends to a triple, and each triple extends to a 4-tuple (of pairwise orthogonal lines). In this sense the configuration ‘tries to be saturated’.

We need more lines to construct a 6-dimensional saturated configuration. Look at the 4-tuples of pairwise orthogonal lines in A . They can be classified. There are tuples of the form

$$Q_1 := \{[1, \bar{1}, 0; 0, 0, 0; 0, 0, 0], \\ [0, 0, 0; 1, \bar{1}, 0; 0, 0, 0], \\ [0, 0, 0; 0, 0, 0; 1, \bar{1}, 0], \\ [1, 1, \bar{2}; 1, 1, \bar{2}; 1, 1, \bar{2}]\},$$

where the bar denotes negation. Similar tuples are obtained by permutations of coordinates. There are 27 tuples of this type.

The other type of tuples is represented by

$$Q_2 := \{[1, \bar{1}, 0; 0, 0, 0; 0, 0, 0], \\ [1, 1, \bar{2}; \bar{2}, 1, 1; \bar{2}, 1, 1], \\ [1, 1, \bar{2}; 1, \bar{2}, 1; 1, \bar{2}, 1], \\ [1, 1, \bar{2}; 1, 1, \bar{2}; 1, 1, \bar{2}]\}.$$

Permutations of coordinates yield 54 different tuples of this form.

The third type of tuples is represented by

$$Q_3 := \{[\bar{2}, 1, 1; \bar{2}, 1, 1; \bar{2}, 1, 1], \\ [\bar{2}, 1, 1; 1, \bar{2}, 1; 1, \bar{2}, 1], \\ [1, \bar{2}, 1; \bar{2}, 1, 1; 1, \bar{2}, 1], \\ [1, \bar{2}, 1; 1, \bar{2}, 1; \bar{2}, 1, 1]\}.$$

Permutations yield again 54 different variants. In total we have $54+54+27 = 135$ ($n_4 = 135$) tuples.

Now look at the subspaces (of the 6-dimensional space R mentioned) *orthogonal* to these 4-tuples. Let ξ, η, ζ be real variables satisfying $\xi + \eta + \zeta = 0$. A generic element of $Q_1^\perp \cap R$ can be written as $[\xi(1, 1, \bar{2}); \eta(1, 1, \bar{2}); \zeta(1, 1, \bar{2})]$. A generic element of $Q_2^\perp \cap R$ is of the form $[0, 0, 0; \xi, \eta, \zeta; -\xi, -\eta, -\zeta]$. The space $Q_3^\perp \cap R$ coincides with $Q_1^\perp \cap R$.

Invoke the exceptional root system G_2 . Its roots are naturally modelled on a 2-dimensional subspace $x + y + z = 0$ of the space of 3-dimensional vectors (x, y, z) . The roots are $(1, \bar{1}, 0), (1, 0, \bar{1}), (0, 1, \bar{1}), (2, \bar{1}, \bar{1}), (\bar{1}, 2, \bar{1}), (\bar{1}, \bar{1}, 2)$, and their inverses (i.e. there are 12 roots). The idea is to identify this subspace with the 2-dimensional subspaces defined by the parameters ξ, η, ζ . In other words, enhance the E_6 projective configuration with the projective lines represented by such vectors, for which (ξ, η, ζ) is an element of G_2 root system. For example, if we take the 4-tuple Q_1 , we obtain six projective lines of the form $[\xi(1, 1, \bar{2}); \eta(1, 1, \bar{2}); \zeta(1, 1, \bar{2})]$, where (ξ, η, ζ) varies over $(1, \bar{1}, 0), (1, 0, \bar{1}), (0, 1, \bar{1}), (2, \bar{1}, \bar{1}), (\bar{1}, 2, \bar{1}), (\bar{1}, \bar{1}, 2)$.

Recall that we have 135 different 4-tuples of pairwise orthogonal projective lines corresponding to the E_6 root system. Construct for each such tuple the six projective lines (invoking the G_2 root system). Take the union of all these lines. This yields 162 lines. Attaching them to A , we obtain a projective configuration $\tilde{A} \supset A$ of 198 elements.

It turns out that \tilde{A} is saturated! We can generate the m -tuples of pairwise orthogonal lines of \tilde{A} . Let \tilde{n}_m be the number of these tuples. The Maple computation results in

$\tilde{n}_2 = 4995, \tilde{n}_3 = 25\,920, \tilde{n}_4 = 32\,400, \tilde{n}_5 = 15\,552, \tilde{n}_6 = 2592$. One could hope that \tilde{A} is non-colourable, but the situation is slightly more complicated. It turns out (see below) that \tilde{A} is colourable, but there is *only one* good bicolouring. This immediately leads to the idea of how to construct a non-colourable configuration containing E_6 . Since two red rays cannot be orthogonal (by the definition of a good bicolouring), one can consider a copy \tilde{A}' of \tilde{A} obtained by some rotation. It is possible to adjust this rotation in such a way that at least one of the red rays in \tilde{A} is orthogonal to a red ray in \tilde{A}' . Then $\tilde{A} \cup \tilde{A}'$ becomes non-colourable. Furthermore (see below), one can choose this rotation in such a way that there exists a saturated finite configuration $\hat{A} \supset \tilde{A} \cup \tilde{A}'$. By that one arrives at a finite saturated non-colourable configuration \hat{A} containing the E_6 configuration.

Let us formulate the final result first and then give some comments. We have a 9-dimensional space \mathbb{R}^9 consisting of vectors $(x; y; z)$, where $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3), z = (z_1, z_2, z_3)$. The symmetric group S_3 naturally acts in four different ways on these vectors: permuting $\{x_i\}_i, \{y_j\}_j, \{z_k\}_k$, or permuting $\{x, y, z\}$. This gives an action of the wreath product $S_3 \wr S_3$ on \mathbb{R}^9 which fixes R (recall that R is defined by the conditions $\sum_i x_i = \sum_j y_j = \sum_k z_k = 0$) on which we realize the E_6 root system. There are also three natural ways to act on \mathbb{R}^9 with the group $\mathbb{Z}/2\mathbb{Z}$ by negating x, y , or z components, respectively. This yields an action of $(\mathbb{Z}/2\mathbb{Z})^3$ on \mathbb{R}^9 , again fixing R . The constructed actions on \mathbb{R}^9 induce the actions on $\mathbb{P}(R)$ (the set of rays in R). For $\lambda \in \mathbb{P}(R)$, denote by $O(\lambda)$ its orbit under the action of $S_3 \wr S_3$, and by $\hat{O}(\lambda)$ the corresponding orbit under the action of the free product of $S_3 \wr S_3$ and $(\mathbb{Z}/2\mathbb{Z})^3$. Consider the following six rays:

$$\begin{aligned} \lambda_1 &:= [1, \bar{1}, 0; \quad 0, 0, 0; \quad 0, 0, 0], \\ \lambda_2 &:= [2, \bar{1}, \bar{1}; \quad 2, \bar{1}, \bar{1}; \quad 2, \bar{1}, \bar{1}], \\ \lambda_3 &:= [1, \bar{1}, 0; \quad 1, \bar{1}, 0; \quad 0, 0, 0], \\ \lambda_4 &:= [2, \bar{1}, \bar{1}; \quad \bar{2}, 1, 1; \quad 0, 0, 0], \\ \lambda_5 &:= [\bar{4}, 2, 2; \quad 2, \bar{1}, \bar{1}; \quad 2, \bar{1}, \bar{1}], \\ \lambda_6 &:= [2, \bar{1}, \bar{1}; \quad 0, 0, 0; \quad 0, 0, 0]. \end{aligned}$$

In this notation we have the following theorem.

Theorem 2.

- (1) The union $A := O(\lambda_1) \cup O(\lambda_2)$ yields the E_6 projective configuration.
- (2) The union $\tilde{A} := \cup_{i=1}^5 O(\lambda_i)$ is a saturated projective configuration, $\tilde{A} \supset A, |\tilde{A}| = 198$. It admits precisely one good bicolouring. The set of red lines coincides with the orbit $O(\lambda_4)$.
- (3) The union $\hat{A} := \cup_{i=1}^6 \hat{O}(\lambda_i)$ is a saturated non-colourable projective configuration, $\hat{A} \supset \tilde{A} \supset A, |\hat{A}| = 558$.

Let us describe the strategy of the implementation of the proof of this theorem on a computer. The facts that \tilde{A} and \hat{A} are saturated can be verified in a straightforward way in Maple. The non-trivial part of the proof is to check that \tilde{A} admits just one good bicolouring.

The set \tilde{A} consists of two disjoint parts: $\tilde{A} = A \cup A_{\text{ext}}$, where A is the set corresponding to E_6 roots. The first step will be to describe *some* (not all) of the elements of $\mathcal{P}_\perp^{(6)}(\tilde{A})$. These elements will be of the shape $T \cup B$, where $T \in \mathcal{P}_\perp^{(4)}(A)$ and $B \in \mathcal{P}_\perp^{(2)}(A_{\text{ext}})$. More precisely, we shall describe a collection of $T_p^{(i)} \in \mathcal{P}_\perp^{(4)}(A)$ and $B_p^{(i)} \in \mathcal{P}_\perp^{(2)}(A_{\text{ext}})$, where $p = 1, 2, \dots, 45$ and $i = 1, 2, 3$, such that for each p one may combine any $T_p^{(l)}$ with any $B_p^{(j)}$ ($l, j = 1, 2, 3$) to obtain $T_p^{(l)} \cup B_p^{(j)} \in \mathcal{P}_\perp^{(6)}(\tilde{A})$.

Observe that there exist 6-tuples (of pairwise orthogonal lines) consisting just of the lines from A_{ext} . For example,

$$\begin{aligned}
 P := \{ & [0, 0, 0; \quad 2, \bar{1}, \bar{1}; \quad \bar{2}, 1, 1], \\
 & [0, 0, 0; \quad 0, 1, \bar{1}; \quad 0, 1, \bar{1}], \\
 & [0, 0, 0; \quad 0, 1, \bar{1}; \quad 0, \bar{1}, 1], \\
 & [\bar{4}, 2, 2; \quad 2, \bar{1}, \bar{1}; \quad 2, \bar{1}, \bar{1}], \\
 & [2, \bar{4}, 2; \quad 2, \bar{1}, \bar{1}; \quad 2, \bar{1}, \bar{1}], \\
 & [2, 2, \bar{4}; \quad 2, \bar{1}, \bar{1}; \quad 2, \bar{1}, \bar{1}]\}.
 \end{aligned}$$

Recall that $|\tilde{A}| = 198$, $|A| = 36$. Therefore $|A_{\text{ext}}| = 162$. Note that 162 is divisible by 6; this is not an accident. Permuting the components of the six vectors (simultaneously), one can generate the 6-tuples similar to P . Each such tuple contains a unique element similar to $[0, 0, 0; 2, \bar{1}, \bar{1}; \bar{2}, 1, 1]$ (i.e. the element in $O(\lambda_4)$). There are 27 tuples obtained this way. It is straightforward to check that they are mutually disjoint. Since $27 \times 6 = 162$, their union is just the set A_{ext} .

Let us number the mentioned 6-tuples of A_{ext} as P_1, P_2, \dots, P_{27} . Each of them contains exactly one point of $O(\lambda_4)$ (represented by the vector with precisely three zero coordinates); denote it by $a_i, i = 1, 2, \dots, 27$. Now look at the A part of $\tilde{A} = A \cup A_{\text{ext}}$. Recall that we have $n_4 = 135$ 4-tuples of pairwise orthogonal lines in A ; number them in some way and denote as T_1, T_2, \dots, T_{135} . For each $i = 1, 2, \dots, 27$, and each $b \in P_i \setminus \{a_i\}$, find all $m, 1 \leq m \leq 135$, such that the elements of T_m are orthogonal to a_i and b . It turns out that every time there are precisely three such 4-tuples. If $m_1 < m_2 < m_3$ are the three numbers of the tuples corresponding to i and b , denote $\Delta_{i,b} := (m_1, m_2, m_3)$.

Now consider the set $D := \{\Delta_{i,b} \mid i = 1, 2, \dots, 27; b \in P_i \setminus \{a_i\}\}$. Its cardinality will be 45. For each $\Delta \in D$ compute all pairs of the form $(i, b), 1 \leq i \leq 27, b \in P_i \setminus \{a_i\}$, such that $\Delta_{i,b} = \Delta$. It turns out that every time there are precisely three such pairs $(i, b), (i', b'), (i'', b'')$ (let $i < i' < i''$); denote $G_\Delta := ((i, b), (i', b'), (i'', b''))$.

Number the elements of $D: \Delta^{(1)}, \Delta^{(2)}, \dots, \Delta^{(45)}$. For each $p = 1, 2, \dots, 45$, let $(m_1^{(p)}, m_2^{(p)}, m_3^{(p)}) = \Delta^{(p)}$; hence for each $p = 1, 2, \dots, 45$, we have three 4-tuples $T_{m_1^{(p)}}, T_{m_2^{(p)}}, T_{m_3^{(p)}}$ consisting of elements of A . From the corresponding $G_{\Delta^{(p)}}$, one obtains the three pairs $(a_i, b), (a_{i'}, b'), (a_{i''}, b'')$ of elements of A_{ext} . Redenote them $(u_p, v_p), (u'_p, v'_p), (u''_p, v''_p)$, respectively. For each $p = 1, 2, \dots, 45$, we have the following data:

$$\sigma_p := (m_1^{(p)}, m_2^{(p)}, m_3^{(p)}; (u_p, v_p), (u'_p, v'_p), (u''_p, v''_p)).$$

Hence, the $27 \times 5 = 135$ distinct pairs of the form $(i, b), i = 1, \dots, 27, b \in P_i \setminus \{a_i\}$ are split into 45 triples $\{(u_p, v_p), (u'_p, v'_p), (u''_p, v''_p)\}$, as well as the collection of 135 distinct 4-tuples T_m is split into 45 triples $\{T_{m_1^{(p)}}, T_{m_2^{(p)}}, T_{m_3^{(p)}}\}$. One may say that each T -triple is attached to a (u, v) -triple in σ_p :

$$\{T_{m_1^{(p)}}, T_{m_2^{(p)}}, T_{m_3^{(p)}}\} \longleftrightarrow \{(u_p, v_p), (u'_p, v'_p), (u''_p, v''_p)\}.$$

For every $p = 1, 2, \dots, 45$, combining a 4-tuple T with any pair (u, v) , one obtains a tuple of six pairwise orthogonal lines in A . For example, $T_{m_3^{(p)}} \cup \{u'_p, v'_p\}$ is a collection of six mutually orthogonal lines. As already mentioned, there exist other 6-tuples of pairwise orthogonal elements in \tilde{A} , but it will suffice to consider just the described ones in order to establish the fact that there exists precisely *one* good bicolouring of the set \tilde{A} .

Now let us make the next step. For each $p = 1, 2, \dots, 45$, using the notation from the definition of σ_p , look at $s_p := \{u_p, u'_p, u''_p\}$. Recall that u_p, u'_p, u''_p are the projective lines of the

form $a_i, 1 \leq i \leq 27$. It turns out that it is possible to partition the set $\{a_1, a_2, \dots, a_{27}\}$ using the sets $\{s_p\}_p$ (note that $27 = 3 \times 9$), i.e. there exists (p_1, p_2, \dots, p_9) such that $s_{p_1}, s_{p_2}, \dots, s_{p_9}$ are pairwise disjoint. As a remark, the Maple computation shows that each s_p is disjoint with precisely 12 other sets of this form. Fix a concrete partition $P = (p_1, p_2, \dots, p_9)$ defined by

$$\begin{aligned} s_{p_1} &= \{[\theta; \xi_1; -\xi_2], [\theta; \xi_2; -\xi_3], [\theta; \xi_3; -\xi_1]\}, \\ s_{p_2} &= \{[\theta; \xi_1; -\xi_1], [\theta; \xi_2; -\xi_2], [\theta; \xi_3; -\xi_3]\}, \\ s_{p_3} &= \{[-\xi_1; \theta; \xi_1], [-\xi_2; \theta; \xi_2], [-\xi_3; \theta; \xi_3]\}, \\ s_{p_4} &= \{[-\xi_2; \theta; \xi_1], [-\xi_3; \theta; \xi_2], [-\xi_1; \theta; \xi_3]\}, \\ s_{p_5} &= \{[\xi_1; -\xi_2; \theta], [\xi_2; -\xi_3; \theta], [\xi_3; -\xi_1; \theta]\}, \\ s_{p_6} &= \{[\xi_1; -\xi_1; \theta], [\xi_2; -\xi_2; \theta], [\xi_3; -\xi_3; \theta]\}, \\ s_{p_7} &= \{[\theta; \xi_1; -\xi_3], [-\xi_2; \theta; \xi_3], [\xi_2; -\xi_1; \theta]\}, \\ s_{p_8} &= \{[\theta; \xi_3; -\xi_2], [-\xi_1; \theta; \xi_2], [\xi_1; -\xi_3; \theta]\}, \\ s_{p_9} &= \{[\theta; \xi_2; -\xi_1], [-\xi_3; \theta; \xi_1], [\xi_3; -\xi_2; \theta]\}. \end{aligned}$$

where θ is as above and $\xi_1 = (\bar{2}, 1, 1), \xi_2 = (1, \bar{2}, 1), \xi_3 = (1, 1, \bar{2})$.

We have a collection of triples $(u_{p_i}, u'_{p_i}, u''_{p_i}), i = 1, 2, \dots, 9$. For each triple we have $(v_{p_i}, v'_{p_i}, v''_{p_i})$ (see the notation in the definition of σ_p); $v_{p_i} \perp u_{p_i}; v'_{p_i} \perp u'_{p_i}; v''_{p_i} \perp u''_{p_i}$. One may also consider $\Delta^{(p_i)} = (m_1^{(p_i)}, m_2^{(p_i)}, m_3^{(p_i)})$. Recall that each $m_j^{(p_i)}$ ($j = 1, 2, 3$) defines a 4-tuple of pairwise orthogonal projective lines in A . Redenote it (this 4-element set) by $T_{i,j}$. A (straightforward) Maple computation shows that it is possible to define $\alpha : \{1, 2, \dots, 9\} \rightarrow \{1, 2, 3\}$ in such a way that the sets $\{T_{i,\alpha(i)}\}_{i=1}^9$ are pairwise disjoint. In fact, there will be just six such functions α ; denote them as $\alpha_1, \alpha_2, \dots, \alpha_6$. For each $k = 1, 2, \dots, 6$, we have nine 4-tuples $\{T_{i,\alpha_k(i)}\}_{i=1}^9$, and each i th tuple can be extended in three different ways up to a 6-tuple (of pairwise orthogonal elements) by way of adjoining $\{u_{p_i}, v_{p_i}\}, \{u'_{p_i}, v'_{p_i}\}$ or $\{u''_{p_i}, v''_{p_i}\}$, respectively.

The third step is to try to implement a good bicolouring. For each $i = 1, 2, \dots, 9$ and $k = 1, 2, \dots, 6$ we have three elements of $\mathcal{P}_\perp^{(6)}(\tilde{A})$ looking as follows: $T_{i,\alpha_k(i)} \cup \{u_{p_i}, v_{p_i}\}, T_{i,\alpha_k(i)} \cup \{u'_{p_i}, v'_{p_i}\}$ and $T_{i,\alpha_k(i)} \cup \{u''_{p_i}, v''_{p_i}\}$. Select any i and k . If one assigns 1 (the red colour) to an element of $T_{i,\alpha_k(i)}$, this implies that all the corresponding u and v lines, as well as the rest of the lines in $T_{i,\alpha_k(i)}$, acquire the assignment 0 (the blue colour). On the other hand, if 1 (the red colour) is assigned to u or v element, say to u_{p_i} , then v_{p_i} becomes blue (is assigned with 0), as well as the four elements of the 4-tuple. The latter implies that one of the elements in $\{u'_{p_i}, v'_{p_i}\}$ and one of the elements in $\{u''_{p_i}, v''_{p_i}\}$ should be red (i.e. have the label 1). In total, one obtains 12 (i.e. $4 + 2^3$) possible choices of colours for the ten elements of

$$c_{i,k} := T_{i,\alpha_k(i)} \cup \{u_{p_i}, v_{p_i}, u'_{p_i}, v'_{p_i}, u''_{p_i}, v''_{p_i}\}.$$

A bicolouring $\chi : \tilde{A} \rightarrow \{0, 1\}$ restricted to $c_{i,k}$ is a map $\chi^{(i,k)} : c_{i,k} \rightarrow \{0, 1\}$. We have 12 candidates for $\chi^{(i,k)}$ in case χ is good; denote them $\chi_1^{(i,k)}, \chi_2^{(i,k)}, \dots, \chi_{12}^{(i,k)}$. The corresponding sets of red lines $R_l^{(i,k)} := \{x \in c_{i,k} \mid \chi_l^{(i,k)}(x) = 1\}, l = 1, 2, \dots, 12$, are either singletons, or 3-element sets. Recall that two red lines cannot be orthogonal. For each $k = 1, 2, \dots, 6$, put $D_k(i_1, l_1; i_2, l_2) := 1$, if $\forall x \in R_{l_1}^{(i_1,k)} \forall y \in R_{l_2}^{(i_2,k)} : y \not\perp x$; put $D_k(i_1, l_1; i_2, l_2) := 0$, otherwise ($i_1, i_2 = 1, 2, \dots, 9; l_1, l_2 = 1, 2, \dots, 12$).

A Maple computation shows that for each pair $(i_1, i_2), i_1 < i_2$, there are 42 pairs (l_1, l_2) such that $D_k(i_1, l_1; i_2, l_2) = 1$. Consider the set L_k consisting of all tuples (l_1, l_2, \dots, l_9) (where each l_m ($m = 1, 2, \dots, 9$) is in the range $1, 2, \dots, 12$), such that

$\forall m, n = 1, 2, \dots, 9 : D_k(m, l_m; n, l_n) = 1$. It turns out (Maple computation) that this set has just five elements, i.e. for each $k = 1, 2, \dots, 6$, there are just five ways to colour the elements of $C_k := \cup_{i=1}^9 c_{i,k}$. Each $l_* \equiv (l_1, l_2, \dots, l_9) \in L_k$ defines a collection of projective lines—a subset $R_k(l_*)$ of $C_k = \cup_{i=1}^9 c_{i,k}$ consisting of elements assigned with 1 (red). The rest are assigned with 0 (blue).

Let us fix at this point our achievements. We have constructed six subsets C_k of \tilde{A} (k varies over $1, 2, \dots, 6$). If $\varkappa : \tilde{A} \rightarrow \{0, 1\}$ is *good*, then we can tell something about the restriction of \varkappa to C_k : we have a limited number of options for $\varkappa|_{C_k}$ indexed by $l_* \in L_k, |L_k| = 5$.

Recall that the set of red rays in C_k corresponding to $l_* \in L_k$ is denoted by $R_k(l_*)$, $k = 1, 2, \dots, 6$. For each k , consider the 9-element set $\tilde{\alpha}_k := \{\alpha_k(i)\}_{i=1}^9$. It turns out (Maple computation) that one can find triples (q_1, q_2, q_3) (let $q_1 < q_2 < q_3$) such that the corresponding $\tilde{\alpha}_{q_1}, \tilde{\alpha}_{q_2}, \tilde{\alpha}_{q_3}$ are pairwise disjoint, i.e. their union has cardinality 27. Furthermore, there will be just two such triples (q_1, q_2, q_3) . In principle, one could now proceed by analysing the sets of the shape $R_{q_1}(l_*^1) \cup R_{q_2}(l_*^2) \cup R_{q_3}(l_*^3)$ where $l_{q_r}^r \in L_{q_r}, r = 1, 2, 3$, but the problem simplifies a little at this stage. One can notice that the sets C_k are invariant with respect to k ranging over $1, 2, \dots, 6$. Furthermore, the set of sets $\{R_k(l_*)\}_{l_* \in L_k}$ is the same for each k ; hence one may denote its five elements by $\mathcal{R}_1, \mathcal{R}_2, \dots, \mathcal{R}_5$. As a remark, two of them consist of 27 elements, and the other three have cardinalities 21.

At this point we can say the following: if $\varkappa : \tilde{A} \rightarrow \{0, 1\}$ is a good bicolouring, then the set of its red rays contains one of $\mathcal{R}_m, m = 1, 2, \dots, 5$.

Now, for each $m = 1, 2, \dots, 5$, consider \mathcal{B}_m consisting of all such lines $x \in \tilde{A}$ (recall that we have them 198), for which there exists $y \in \mathcal{R}_m$ such that $y \perp x$. If \mathcal{R}_m is contained in the set of red rays of \varkappa , then all elements of \mathcal{B}_m must be blue (by the definition of good bicolouring). Now look at $\mathcal{P}_\perp^{(6)}(\tilde{A})$. If $\mathcal{B}_m, m = 1, 2, \dots, 5$, contains at least one of these 6-tuples, the corresponding variant with m should be ruled out. It turns out (Maple computation) that just one of the five variants survives after the verification of this condition. Denote the set \mathcal{R}_m corresponding to this unique m by $\tilde{\mathcal{R}}$. From a Maple computation we obtain that $\tilde{\mathcal{R}}$ consist of 27 lines of the shape $[0, 0, 0; 2, \bar{1}, \bar{1}; \bar{2}, 1, 1]$, i.e. those which are represented by a vector with precisely three zeros. In the notation of the theorem, this is just the orbit $O(\lambda_4)$. It remains to check that if we colour all rays from $\tilde{\mathcal{R}}$ to red, and all other rays in \tilde{A} to blue, then the conditions of the definition of a good bicolouring are satisfied. This yields the *unique* good bicolouring of \tilde{A} .

Now let us consider the set $\hat{A} \supset \tilde{A}$ from the theorem. As a side effect of the computations, the numbers \hat{n}_k of elements in $\mathcal{P}_\perp^{(k)}(\hat{A}), k = 2, 3, \dots, 6$ are as follows: $\hat{n}_2 = 18\,423, \hat{n}_3 = 104\,978, \hat{n}_4 = 136\,620, \hat{n}_5 = 66\,744, \hat{n}_6 = 11\,124$. For each $l = [x_1, x_2, x_3; y_1, y_2, y_3; z_1, z_2, z_3] \in \hat{A}$, construct a ray $l_1 := [-x_1, -x_2, -x_3; y_1, y_2, y_3; z_1, z_2, z_3]$. Denote the union of all l_1 by \tilde{A}_1 . Similarly define the sets A_2 and A_3 as the unions of all l_2 of the form $l_2 := [x_1, x_2, x_3; -y_1, -y_2, -y_3; z_1, z_2, z_3]$, and all $l_3 := [x_1, x_2, x_3; y_1, y_2, y_3; -z_1, -z_2, -z_3]$, respectively.

We have $\hat{A} \supset A, A_1, A_2, A_3$. Suppose that \hat{A} admits a good bicolouring $\hat{\varkappa}$. The red subset $\tilde{\mathcal{R}}$ of \tilde{A} is known (the 27 elements of the orbit $O(\lambda_4)$). Consider three reflections $P_1 : [x; y, z] \mapsto [-x; y; z], P_2 : [x; y, z] \mapsto [x; -y; z], P_3 : [x; y, z] \mapsto [x; y; -z]$. The red subsets of \hat{A}_i (to be denoted as $\tilde{\mathcal{R}}_i, i = 1, 2, 3$), are obtained by applying these reflections to $\tilde{\mathcal{R}}$. Put $\hat{\mathcal{R}} := \tilde{\mathcal{R}} \cup \tilde{\mathcal{R}}_1 \cup \tilde{\mathcal{R}}_2 \cup \tilde{\mathcal{R}}_3$. The cardinality of $\hat{\mathcal{R}}$ will be 54. The set of red rays of $\hat{\varkappa}$ should contain $\hat{\mathcal{R}}$. The non-colourability of \hat{A} is derived now from the following fact (checked in Maple): there exists a 6-tuple from $\mathcal{P}_\perp^{(6)}(\hat{A})$ such that the cardinality of its intersection with $\hat{\mathcal{R}}$ is not equal to 1, i.e. either we obtain a completely blue 6-tuple of pairwise orthogonal lines or encounter a situation when several red lines are mutually orthogonal. This contradicts the assumption that $\hat{\varkappa}$ is good. Therefore \hat{A} is non-colourable.

5. Discussion

It is interesting to mention that the notion of a saturated projective configuration is intimately related with the notion of an orthoalgebra. An *orthoalgebra* is a set S equipped with a relation $\perp \subset S \times S$, a map $\cdot \oplus \cdot : \perp \rightarrow S$, $(x, y) \mapsto x \oplus y$, and two distinct elements $\mathbf{0}, \mathbf{1} \in S$; these data satisfy (1) if $x \oplus y$ is defined, then $x \oplus y = y \oplus x$; (2) if $(x \oplus y) \oplus z$ is defined, then $(x \oplus y) \oplus z = x \oplus (y \oplus z)$; (3) $x \oplus \mathbf{0}$ is always defined and $x \oplus \mathbf{0} = x$; (4) $\forall x \exists ! x^* : x \oplus x^* = \mathbf{1}$; (5) if $x \oplus x$ is defined, then $x = \mathbf{0}$. A prototypical example of an orthoalgebra is the Hilbert space orthoalgebra $\mathbb{L}(\mathcal{H})$: the set S is the set of all subspaces of the Hilbert space \mathcal{H} , and \oplus is the orthogonal sum.

If A is a finite saturated projective configuration in \mathcal{H} , then it generates a finite suborthoalgebra of $\mathbb{L}(\mathcal{H})$. The examples of such configurations are given above, but there exist others, for instance, [2, 9]. Note that the corresponding partial Boolean algebra (see [8]), need not be finite. Orthoalgebras attract attention as the structures capturing the logic of quantum theory [10, 11]. The relation between ‘quantum logic’ and Kochen–Specker-type constructions (i.e. non-bicolourable finite configurations) is discussed in [12, 13] and in [14, 15]. If the finite saturated configuration A is non-bicolourable (i.e. is of Kochen–Specker type), then this fact is translated into the absence of a morphism from the corresponding orthoalgebra to a two-element orthoalgebra (absence of bivaluations). A series of examples of such orthoalgebras has been constructed in [16]; in particular, the orthoalgebra corresponding to the configuration described in [6] is isomorphic to the E_8 orthoalgebra.

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